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On the Relationship between Parsimonious Covering and Boolean Minimization

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Abstract

Minimization of Boolean switching functions is a basic problem in the design of logic circuits. The designer first comes up with a switching function expressed in terms of several binary input variables that satisfies the desired functionality, and then attempts to minimize the function as a *sum of products* or *product of sums*. It turns out that a sum of products form of a switching function that has no redundancy is a union of *prime implicants* of the function.

In this paper we would like to explicate some of the relationships of the boolean minimization problem to a formalization of *abductive inference* called *parsimonious covering*. Abductive inference often occurs in diagnostic problems such as finding the causes of circuit faults [Reiter, 87] or determining the diseases causing the symptoms reported by a patient [Peng and Reggia, 90]. Parsimonious covering involves covering all observed facts by means of a parsimonious set of explanations that can account for the observations. The relationship of parsimonious covering to boolean minimization has been noted by the developers of the theory; we intend to pursue a detailed mapping here.

1 Introduction

In Boolean minimization, one is interested in minimizing the number of terms (possibly with fewer literals) in the expression of a Boolean switching function [Kohavi, 78]. In other words, the goal is to find an expression for a minimal cost logic circuit that causes the same functionality as the given boolean function. This goal is very similar to the *abductive* goals of explaining the faults in circuits or diagnosing a set of medical symptoms in a patient. One of the computer models for certain classes of abductive inference is *parsimonious covering*. This theory has been developed for diagnostic problem solving [Peng and Reggia, 90], but has also been extended to other domains such as language processing [Dasigi, 89].

Parsimonious covering involves covering or accounting for the set of observed manifestations using a parsimonious set of possi-

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ble causes. Once again, this is very similar to covering the desired switching function using a minimal or irredundant set of terms as is the goal of Boolean minimization. Not surprisingly, *minimality* and *irredundancy* have been used as criteria of parsimony in parsimonious covering theory. Several of these relationships have been noted by the developers of parsimonious covering, but have never been explicated before, to our knowledge. We pursue a detailed mapping here, and observe that features of either problem may be captured in terms of those of the other. Thus, we hope this paper will be of interest to either community. After quick reviews of Boolean minimization and parsimonious covering, we show how they capture each other's features, and conclude with our plans for further work.

2 Boolean Minimization Problem

A brief review of the Boolean Minimization problem (henceforth referred to as BMP) follows [Kohavi, 78]. The boolean *constants* true and false are denoted as 0 and 1; the boolean operations and, or and not as $\wedge$, $\vee$ and $\neg$; and boolean *variables* as $x_0, x_1, \ldots, x_n$. Any boolean constant or variable is a *boolean expression*, and if B_1 and B_2 are boolean expressions, then so are $\neg B_1$, $\neg B_2$, $B_1 \vee B_2$ and $B_1 \wedge B_2$. x and $\neg x$ are called *literals*. A conjunction (respectively, disjunction) of literals is called a *product term* (resp. *sum term*). A product term (resp. sum term) containing literals involving *all* input variables is called a *minterm* (resp. *maxterm*). A boolean expression is in *sum-of-products form* (resp. *product-of-sums form*) if it is expressed as a disjunction (resp. conjunction) of product terms (resp. sum terms). Two boolean expressions B_1 and B_2 containing variables $\{x_0, x_1, \ldots, x_n\}$ are said to be *logically equivalent* if B_1 and B_2 have the same values for all possible combinations of values of the variables $\{x_0, x_1, \ldots, x_n\}$. Corresponding to every boolean expression there exist logically equivalent boolean expressions that are in sum-of-products (resp. product-of-sums) form.

A sum-of-products expression is *minimal* if there is no other expression with smaller number of product terms and with fewer literals. A sum-of-products expression is *irredundant* if it is not possible to delete a product term or a literal from it without altering its logical value. A minimal expression is not always unique, but is always irredundant. However, an irredundant expression may not necessarily be minimal.

An *implicant* of a boolean expression is a product term that logically implies it. A *prime implicant* of a boolean expression is an implicant that does not logically imply any other implicant of the expression. An *essential prime implicant* is a prime implicant that does not logically imply any disjunction of other prime implicants.

A boolean expression $f(x_0, x_1, \ldots, x_n)$ represents a *monotonic* function, if it satisfies the following condition:

$$\forall i \leq n : [f(x_0, \ldots, x_i := 0, \ldots, x_n) = 1]$$
$$\Rightarrow [f(x_0, \ldots, x_i := 1, \ldots, x_n) = 1]$$

3 Parsimonious Covering Theory

A brief review of Parsimonious Covering Problem (henceforth referred to as PCP) follows [Peng and Reggia, 90]. A *diagnostic*

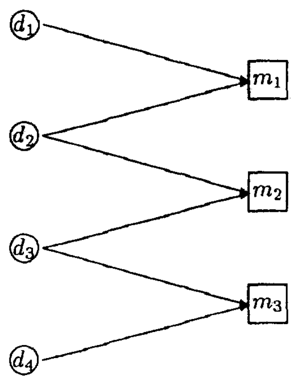

Figure 1: Parsimonious Covering Problem

problem P is a 4-tuple $\langle D, M, C, M^+\rangle$ where D is a finite set of disorders; M is a finite set of manifestations; $C \subseteq D \times M$ is the causation relation; and $M^+ \subseteq M$ is the set of observed manifestations. For any $d_i \in D$ and $m_j \in M$, $effects(d_i) = \{m_j \mid \langle d_i, m_j\rangle \in C\}$ and $causes(m_j) = \{d_i \mid \langle d_i, m_j\rangle \in C\}$. For any $D_I \subseteq D$, $effects(D_I) = \{m_j \mid \langle d_i, m_j\rangle \in C \wedge d_i \in D_I\}$. The set $D_I \subseteq D$ is said to be a *cover* of $M_J \subseteq M$ if $M_J \subseteq effects(D_I)$.

A set $E \subseteq D$ is said to be an *explanation* of M^+ for a problem $P = \langle D, M, C, M^+\rangle$ iff E covers M^+ and E satisfies a given parsimony criterion. A cover D_I of M_J is said to be *minimal* if its cardinality is smallest among all covers of M_J. A cover D_I of M_J is said to be *irredundant* if none of its proper subsets is also a cover of M_J. A minimal cover is irredundant, but the converse does not hold.

We illustrate the above definitions with an example depicted in Figure 1. $\{d_1, d_2, d_3, d_4\}$ are the diseases, $\{m_1, m_2, m_3\}$ are the manifestations. $\{d_1, d_2\}$ is a cover of $\{m_1, m_2\}$, while $\{d_2\}$ is the minimum cover of $\{m_1, m_2\}$. $\{d_1, d_3\}$ and $\{d_2, d_4\}$ are two minimal covers of $\{m_1, m_2, m_3\}$. $\{d_3\}$ and $\{d_2, d_4\}$ are both irredundant covers of $\{m_2, m_3\}$; the former one is minimal, while the latter one is not. Algorithms for computing minimal and irredundant covers have been developed and have been extended to more complex knowledge structures involving chains of causal links (e.g., an overheated resistor may cause a nearby transistor to malfunction, which, in turn, may change the output of a gate).

4 Encoding a PCP as a BMP

We now show that an instance of PCP can be encoded as an instance of BMP. The set of diseases of PCP is the set of boolean variables of BMP. Covering a manifestation m requires one of the $causes(m)$ to be present. This is equivalent to saying that the disjunction of d's in $causes(m)$ be true. To cover M^+, the set of observed manifestations, we need to cover each manifestation in M^+. Thus, the boolean expression obtained by conjoining the aforementioned disjunctions (of causative diseases) for each observed manifestation should be true. The following result specifies the relationship between PCP and its encoding as a BMP.

Lemma 1 *Given an instance of the diagnostic problem $P = \langle D, M, C, M^+\rangle$, the corresponding instance of the boolean minimization is the product-of-sums expression $B_{ps} = \bigwedge\{\bigvee\{d_i \mid (d_i, m_j) \in C\} \mid m_j \in M^+\}$, where the set D corresponds to the set of boolean variables. Let B_{sp} be the corresponding sum-of-products expression obtained by distributing $\wedge$s over $\vee$s. Then, every implicant of B_{sp} corresponds to a cover of M^+; every prime implicant of B_{sp} to an* *irredundant cover of M^+; and every prime implicant B_{sp} with the least number of literals, to a minimal cover of M^+.*

We illustrate this with the example shown in Figure 1. If $M^+ = \{m_2, m_3\}$, then the PCP instance can be encoded as: $(d_2 \vee d_3) \wedge (d_3 \vee d_4)$. A logically equivalent sum-of-products form is:

$$d_2 \wedge d_3 \ \vee \ d_2 \wedge d_4 \ \vee \ d_3 \ \vee \ d_3 \wedge d_4.$$

The prime implicants are d_3 and $(d_2 \wedge d_4)$. Only d_3 qualifies to be minimal (and hence, irredundant), while $d_2 \wedge d_4$ is irredundant (but not minimal).

In general, minimal expressions are not unique. For instance, both

$$\neg x_1 \wedge \neg x_3 \ \vee \ x_1 \wedge \neg x_2 \ \vee \ x_2 \wedge x_3$$

$$\neg x_1 \wedge x_2 \ \vee \ \neg x_2 \wedge \neg x_3 \ \vee \ x_1 x_3$$

are logically equivalent distinct minimal expressions. On the other hand, observe that the Boolean expressions encoding a PCP does not contain the negation operator. Thus, it can be shown that

Lemma 2 *The boolean expression arising as the encoding of the PCP represents a monotonic function, which can be represented by a unique minimal expression in sum-of-products form.*

This lemma is the basis of the following algorithm to compute *all* irredundant and minimal covers. Given an instance of a PCP, encode it as a BMP, and transform the expression so obtained into a logically equivalent expression in sum-of-products form by distributing $\wedge$s over $\vee$s. Each disjunct is an implicant of the original expression. The set of implicants so obtained is then partitioned into groups $g_1, g_2, \ldots, g_n$, where g_i contains all implicants with i distinct literals. Starting from group g_2, delete implicants, from all groups g_i, that logically imply other implicants that appear in groups with lower index. This procedure terminates leaving only prime implicants, which constitute the set of irredundant covers. The members of the nonempty group g_i with the least index i constitute the set of minimal covers.

5 Encoding a BMP as a PCP

Parsimonious covering was originally conceived as a formal model of the way diagnostic inferences are performed by human diagnosticians. The first version of the theory was a generalization of the set covering problem of mathematics [Edwards, 62]. Parsimonious covering has found applications in error classification in some discrete sequential processes [Ahuja, 85], software engineering [Basili and Ramsey, 85], growth models of biological tree structures [Tagamets and Reggia, 85], treatment selection [Neapolitan, et al., 87] and natural language processing [Dasigi, 89]. ¿From this perspective, it is interesting to see yet another application in Boolean minimization for parsimonious covering.

The problem of minimizing a Boolean expression essentially consists of two major steps: that of determining all prime implicants of the function and that of selecting a minimal (or irredundant) subset of prime implicants that can cover all the minterms of the given boolean function. Now, it is obvious that there exists a straightforward mapping between concepts underlying parsimonious covering (especially in the context of diagnostic problems) and those un-

Boolean Minimization	Parsimonious Covering
minterms	manifestations
prime implicants	disorders
implication	causal relation
x is implied by $x \wedge \neg y$	d causes m
x covers $x \wedge \neg y$	d covers m
boolean function to be minimized	observed manifestations
minimized form of boolean function	minimal cover
irredundant set of prime implicants	irredudant cover
don't cares	manifestations that may or may not be covered
essential prime implicants	disorders covering pathognomonic manifestations

Table 1: Mapping between Parsimonious Covering and Boolean Minimization

derlying Boolean minimization. We summarize several such relationships in Table 1.

In a BMP, a set of minterms represent a boolean expression to be minimized, analogous to a set of observed manifestations to be explained in a PCP. The minimized form of a boolean expression always consists of prime implicants, just as disorders are used to explain manifestations. A prime implicant is said to cover all the minterms that imply it, which is analogous to a disorder's potential to cover all the manifestations it can cause. In a BMP, a prime implicant may be implied by don't care minterms. Similarly, in a specific PCP, a disorder need not necessarily cause all manifestations it can potentially cause. Any minimized form of a switching function *must* consist of essential prime implicants. The analogous concept in a PCP would be disorders that uniquely correspond to pathognomonic manifestations. Finally, in a typical BMP, only the minimized form of a boolean function may be of interest, while in a PCP, one can talk about minimal covers as well as irredundant covers. In diagnostic problems, the latter type of covers appear to be so interesting that they are often called (syntactically) minimal covers.

6 Conclusions

¿From this preliminary work, we draw the following conclusions, which also indicate directions for further work:

- It may be noted that *only* the prime implicants of a given boolean function in a BMP, rather than any general product terms, are considered analogous to disorders in a PCP. If any general product term were treated as the BMP-analog of a disorder, then there would be a trivial minimum cover, namely, 1, that can "cover" (in a different sense) any boolean function. The specific choice of prime implicants as the BMP-analog of disorders captures the important notion of logical equivalence of the minimal cover to the original boolean expression.

- As already mentioned, BMP consists of two major steps. An interesting observation in this context is that in Section 4, a PCP has been mapped into the first step of a BMP (the determination of all prime implicants), while in Section 5, the second step of a BMP (the selection of a minimal or irredundant subset of prime implicants that can cover the given boolean function) has been mapped into a PCP. This suggests the possibility that the complete BMP may be equivalent to the PCP. After all, parsimony (lack of redundancy) is a notion germane to prime implcants.

- It does seem possible to start with a boolean function expressed as the sum of several minterms (product terms involving *all* input variables) and cover it with product terms involving one fewer variable that are implicants of the function, and then with still smaller product terms, etc. Each step of covering in this process roughly corresponds to what Peng and Reggia call *layers* (more precisely, ***pseudo-layers***) [Peng and Reggia, 90]. We hope that work in this direction identifies closer ties between the two formalisms and also that theoretical results in each area translate into equivalent results in the other. In particular, it would be interesting to see if transitivity of complete sets of irredundant covers applies to analogous notions in the BMP.

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